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Exploring the Conformal Constraint Equations

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1 Introduction

A model for the asymptotic structure of spacetime was suggested by Roger Penrose and others [**?**], see also [**?**, **?**], using the technique of conformal rescaling. Since the reader is by now familiar with the details of the conformal rescaling construction, only enough will be said here to fix the notation to be used in the remainder of this article. The object under study will consist of a *physcal spacetime* $\tilde{M}$ with Lorentz metric $\tilde{g}$ satisfying Einstein's equation $Ric(\tilde{g}) = 0$ (for simplicity, only the vacuum equations will be considered here), that is conformally diffeomorphic to the interior of an *unphysical spacetime*, a Lorentz manifold M with boundary. Denote by g the Lorentz metric of M and by Ω the conformal factor. Recall that the conformal factor is also a defining function for ∂M; that is, $\Omega = 0$ only on ∂M but such that $\mathrm{d}\Omega \neq 0$ at any point of ∂M. The conformal equivalence between $\tilde{M}$ and M requires that $\tilde{g} = \Omega^{-2} g$ on the interior of M (the pull-back of the conformal diffeomorphism has been suppressed for convenience).

The boundary ∂M is known as *future null infinity* because it can be shown that all null geodesics have their future endpoints lying on ∂M. The asymptotic properties of the physical spacetime in null directions can thus be examined by studying the properties of the unphysical spacetime near its boundary. To this end, one uses the fact that, by virtue of the conformal equivalence with the physical spacetime, the quantities Ω and g satisfy a conformal version of Einstein's equation, namely that $Ric(\Omega^{-2} g) = 0$. However, this equation has the significant deficiency that it is degenerate near the boundary of M because there $\Omega \to 0$, and is thus not ideally suited for analytic investigations of the nature of the spacetime at null infinity. One possible solution of this difficulty is to use a technique developed by Friedrich [**?**], which aims to describe the geometry of the unphysical spacetime by means of a new system of equations derived from the equation $Ric(\Omega^{-2} g) = 0$ that is fully equivalent to this equation but is formally regular at the boundary of the unphysical spacetime. These equations involve g, Ω and several other quantities and are known as the *conformal Einstein equations.*

As with Einstein's equation in the physical spacetime, one can formally attempt to solve the unphysical equations by means of an initial value formulation, where appropriate initial data is defined on an asymptotically *hyperboloidal* hypersurface, i. e. a spacelike hypersurface $\mathcal{Z}$ that intersects ∂M transversely in such a way that $\partial \mathcal{Z}$ is diffeomorphic to $\mathbf{S}^2$ and equals $\mathcal{Z} \cap \partial M$. As in the usual initial value formulation of Einstein's equations in the physical spacetime, the initial data must satisfy certain equations on $\mathcal{Z}$. These equations are known as the *conformal constraint equations.*

The conformal constraint equations form a complicated system of coupled, nonlinear differential equations, and the purpose of this paper is to introduce these equations and to investigate some of their properties in order to set up a perturbative approach for generating solutions in the neighbourhood of a known solution. A solution of these inquations in their full generality using these techniques remains, as yet, out of reach. However, the perturbative method is successful in a certain, preliminary case, and this will be presented herein. To be precise, the following theorem will be proved.

Main Theorem: *The conformal constraint equations, under appropriate assumptions, become*

$$\begin{aligned} Ric(h) &= S \\ \mathrm{div}(S) &= 0\,, \end{aligned} \tag{1}$$

on $\mathbf{R}^3$*, where* h *is the metric of* $\mathbf{R}^3$ *and* S *is a symmetric 2-tensor that is trace free with respect to* h*. There exists an infinite dimensional space of solutions* (h, S) *that is* near $(\delta, 0)$ *(*δ *is the Euclidean metric of* $\mathbf{R}^3$*) in a suitably defined topology.*

The existence of solutions near $(\delta, 0)$ will be found by applying an Implicit Function Theorem argument, based on certain standard techniques for handling the operators appearing in (1). However, these techniques will be shown to be slightly inadequate for dealing with the complexities of (1) and thus suitable modifications will be made to the standard techniques which succeed in generating solutions.

2 The Conformal Constraint Equations

2.1 Deriving the Equations

Suppose (M, g, Ω) is an unphysical spacetime and that the metric and conformal factor satisfy

$$Ric(\Omega^{-2} g) = 0\,. \tag{2}$$

This section sketches briefly how this equation for Ω and g leads first to the conformal Einstein equations and then to the conformal constraint equations. Begin by expanding the Ricci curvature equation (2) to obtain

$$R_{\mu\nu} = -\frac{\Box\Omega}{\Omega}\, g_{\mu\nu} - \frac{2}{\Omega}\, \nabla_\mu \nabla_\nu \Omega + \frac{3\,\|\nabla\Omega\|^2}{\Omega^2}\, g_{\mu\nu}\,, \tag{3}$$

where $R_{\mu\nu}$ are the components of the Ricci tensor in the unphysical spacetime, ∇_μ is the covariant derivative of the four-metric and $\Box$ is the D'Alembertian operator of the four-metric. Notice that, as it is written, equation (3) contains Ω^{-1} terms which tend to infinity near the boundary of the unphysical spacetime. Alternatively, if the equation is multiplied through by Ω^2, then the principal part of the differential operator acting on g, contained in the second-order expression $R_{\mu\nu}(g)$, would tend to zero there. Either way, equation (3) degenerates near the boundary of the unphysical spacetime, making this an unwieldy choice for studying the geometry of the spacetime near null infinity. Friedrich proceeded in the following way in order to avoid this difficulty. Let $C_{\mu\nu\lambda\rho}$ be the Weyl tensor of the metric g and define the quantities

$$\begin{aligned} L_{\mu\nu} &= \frac{1}{2} R_{\mu\nu} - \frac{1}{12} R g_{\mu\nu} \\ S_{\mu\nu\lambda\rho} &= \Omega^{-1} C_{\mu\nu\lambda\rho} \\ \psi &= \frac{1}{4}\Delta\Omega + \frac{1}{24} R\Omega\,. \end{aligned} \tag{4}$$

It can be shown that the Weyl tensor of g vanishes at the boundary of M [**?**], ensuring that the tensor $S_{\mu\nu\lambda\rho}$ is smooth on ∂M. Friedrich then found that the system of equations

$$\begin{aligned} \nabla_\mu \nabla_\nu \Omega &= -\Omega L_{\mu\nu} + \psi g_{\mu\nu} \\ \nabla_\mu \psi &= -L_{\mu\nu} \nabla^\nu \Omega \\ \nabla_\lambda L_{\mu\nu} - \nabla_\mu L_{\lambda\nu} &= \nabla^\rho S_{\mu\lambda\nu\rho} \\ \nabla^\rho S_{\mu\lambda\nu\rho} &= 0 \\ 2\Omega\psi - \nabla_\mu \Omega \nabla^\mu \Omega &= 0 \\ R_{\mu\nu\lambda\rho} &= \Omega S_{\mu\nu\lambda\rho} + g_{\mu[\lambda} L_{\nu]\rho} - L_{\mu[\lambda} g_{\nu]\rho}\,, \end{aligned} \tag{5}$$

where $R_{\mu\nu\lambda\rho}$ is the Riemann curvature tensor of the metric g, can be derived from (3) as well as the Bianchi identity and the decomposition of the curvature tensor given by the last line of (5); these are the conformal Einstein equations. Furthermore, he showed that these new equations are *equivalent* to (3) when the quantities L, S and ψ as well as g and Ω are considered as the unknowns in the system of equations (5): if (g, Ω, L, S, ψ) satisfies (5), then the pair (g, Ω) satisfies (3) and that L, S and ψ relate to Ω and curvature quantities in the manner indicated in (4). The essential difference between the conformal Einstein equations and the usual Einstein equations, expressed for a metric of the form $\Omega^{-2}g$ as in (3), is that they remain formally regular as $\Omega \to 0$ near the boundary of M.

Suppose now that $\mathcal{Z}$ is an asymptotically hyperboloidal hypersurface in M. The fact that the conformal Einstein equations constrain the initial data on $\mathcal{Z}$ can be seen by performing a $3+1$ splitting of spacetime near $\mathcal{Z}$. Choose a frame E_a, $a = 1, 2, 3$, for the tangent space of $\mathcal{Z}$ and complete this to a frame for the unphysical spacetime by adjoining the forward-pointing unit normal vector field n of $\mathcal{Z}$ (assume that M is time-oriented). The quantities g, Ω, L, S and ψ now induce the following list of initial data on $\mathcal{Z}$:

- the induced metric h of $\mathcal{Z}$
- the second fundamental form χ of $\mathcal{Z}$
- the function Ω restricted to $\mathcal{Z}$
- the normal derivative $n(\Omega)\big|_{\mathcal{Z}}$, to be denoted Σ
- the tensors $L_{ab} = E^\mu_a E^\nu_b L_{\mu\nu}$ and $L_a = n^\mu E^\nu_a L_{\mu\nu}$
- the tensors $S_{abc} = n^\mu E^\nu_a E^\lambda_b E^\rho_c S_{\nu\mu\lambda\rho}$ and $S_{ab} = n^\mu n^\nu E^\lambda_a E^\rho_b S_{\lambda\mu\rho\nu}$
- and the function ψ restricted to $\mathcal{Z}$.

These quantities are those which appear in (5) in equations that do not contain any second normal derivatives of g or Ω, or first normal derivatives of L, S or ψ when they are restricted to $\mathcal{Z}$ and decomposed using the $3+1$ splitting given by the frame $\{n, E_a\}$. The constraint equations which arise in this manner are the conformal constraint equation and are given by

$$
\begin{aligned}
D_a D_b \Omega &= \Sigma \chi_{ab} - \Omega L_{ab} + \psi h_{ab} \\
D_a \Sigma &= \chi^c_a D_c \Omega - \Omega L_a \\
D_a \psi &= -D^b \Omega L_{ba} - \Sigma L_a \\
D_a L_{bc} - D_b L_{ac} &= D^e \Omega S_{ecab} - \Sigma S_{cab} - (\chi_{ac} L_b - \chi_{bc} L_a) \\
D_a L_b - D_b L_a &= D^e \Omega \, S_{eab} + \chi^c_a L_{bc} - \chi^c_b L_{ac} \\
D^a S_{abc} &= \chi^a_b S_{ac} - \chi^a_c S_{ab} \\
D^a S_{ab} &= -\chi^{ac} S_{abc} \\
D_c \chi_{ba} - D_b \chi_{ca} &= \Omega S_{abc} + h_{ab} L_c - h_{ac} L_b \\
r_{ab} &= \Omega S_{ab} + L_{ab} + \frac{1}{4} L^c_c h_{ab} - \chi^c_c \chi_{ab} + \chi_{ca} \chi^c_b \\
0 &= 2\Omega\psi + \Sigma^2 - \|D\Omega\|^2
\end{aligned}
\tag{6}
$$

where D denotes the covariant derivative operator on $\mathcal{Z}$ corresponding to its induced metric h. The various tensor quantities that appear in (6) possess certain symmetries as a result of their origin as components of the curvature tensor: L_{ab} is symmetric; S_{ab} is symmetric and trace free; and S_{abc} is antisymmetric on its last two indices, satisfies the Jacobi symmetry $S_{abc} + S_{cab} + S_{bca} = 0$ and is trace free on all its indices. (Tensors with these symmetries will appear often in the sequel. Tensors of rank three that are antisymmetric on their last two indices and satsify the Jacobi symmetry will be called Jacobi tensors for short while those which are in addition trace-free will be called traceless Jacobi tensors.) Note that even though the tensor $S_{abcd} = E_a^\mu E_b^\nu E_c^\lambda E_d^\rho S_{\mu\nu\lambda\rho}$ appears in the constraint equations, it is not a truly independent initial datum because, thanks to the symmetries of $S_{\mu\nu\lambda\rho}$, it can be written as $S_{abcd} = h_{a[c}S_{d]b} - S_{a[c}h_{d]b}$.

The system (6) is clearly exceedingly complicated, if only because of its large profusion of variables and equations. Further complications come from its high degree of coupling and its non-ellipticity. However, the advantage provided by (6) is once again that it is formally regular at the boundary of $\mathcal{Z}$. For comparison, it is worthwhile looking at the usual constraint equations. The interior of $\mathcal{Z}$ is diffeomorphic to a spacelike hypersurface $\tilde{\mathcal{Z}}$ in the physical spacetime $\tilde{M}$ which possesses an induced metric $\tilde{h}$ and a second fundamental form $\tilde{\chi}$. By virtue of Einstein's equation in the physical spacetime, $\tilde{h}$ and $\tilde{\chi}$ satisfy the constraint equations

$$\begin{aligned} \tilde{D}^a \tilde{\chi}_{ab} - \tilde{D}_b \tilde{\chi}_a^a &= 0 \\ \tilde{R} - (\tilde{\chi}_a^a)^2 + \tilde{\chi}^{ab}\tilde{\chi}_{ab} &= 0\,, \end{aligned} \tag{7}$$

where $\tilde{D}$ is the covariant derivative operator of the metric $\tilde{h}$ and $\tilde{R}$ is its scalar curvature. These equations can be rephrased in terms of h, χ and Ω in the unphysical spacetime by conformal transformation. The necessary transformation rules are, of course, that $\tilde{h} = \Omega^{-2}h$ and also that $\tilde{\chi} = \Omega^{-1}\chi + \Sigma\Omega^{-2}h$ (which can be found by conformally transforming the definition of the second fundamental form as the normal component of the covariant derivative restricted to $\tilde{\mathcal{Z}}$). The resulting equations are

$$\begin{aligned} \Omega^2\big(R - (\chi_a^a)^2 + \chi^{ab}\chi_{ab}\big) + 6\Omega\Delta\Omega - 12\|D\Omega\|^2 + 4\Omega\Sigma\chi_a^a + 6\Sigma^2 &= 0 \\ \Omega\big(D_a\chi_b^a - D_b\chi_a^a\big) - 2D_b\Sigma - 2\chi_b^a\, D_a\Omega &= 0\,, \end{aligned} \tag{8}$$

where $\Sigma = n(\Omega)\big|_{\mathcal{Z}}$ and Δ is the Laplacian of the metric h. One sees once again that an equations arises whose principal parts are contain factors of Ω, and thus degenerate as $\Omega \to 0$ near the boundary of $\mathcal{Z}$.

The conformal constraint equations listed in (6) are fully equivalent to the usual constraint equations (8), in the sense that if $(h, \chi, \Omega, \Sigma)$ solves (8) and the subsidiary quantities S, L and ψ are defined as indicated in (6) (e. g. the last equation defines ψ; then the first equation defines the 2-tensor L_{ab}, etc.), then the conformal constraint equations are satisfied. Furthermore, if $(h, \chi, \Omega, \Sigma, S, L, \psi)$ satisfies (6), then $(h, \chi, \Omega, \Sigma)$ satisfies (8); and consequently, $\tilde{h}$ and $\tilde{\chi}$, given by the transformation rules above, satisfy the usual constraint equations (7). One method for constructing solutions of the conformal constraint equations is thus obvious: construct any solution $(\tilde{h}, \tilde{\chi})$ of the usual constraint equations, choose a conformal factor, perform the transformations to the unphysical spacetime and use the conformal constraint equations to define the subsidiary quantities in terms of $(\tilde{h}, \tilde{\chi})$. Then these new quantities satisfy the conformal constraint equations. This method was used in [**?**], where the small time evolution of the initial data was studied using the conformal Einstein equations. However, because of the problem of the vanishing of the conformal factor near the boundary of the unphysical spacetime and the resultant degeneration of the equation (8), this method is not suitable for understanding or controlling the behaviour of the solution near null infinity. It is for this reason that new methods for solving (6) directly must be developed.

The complexity of the conformal constraint equations makes this a daunting task. However, a great deal of structure is contained within these equations, and the hope is that this structure can be exploited in the search for methods of obtaining solutions. However, before aspiring to this long-term goal which is as yet beyond the scope of this article, it is worthwhile to consider a reduction of the conformal constraint equations to a certain special case where some of the important features appear in isolation and can be tackled more easily.

2.2 Reduction to the Extended Constraint Equations

The special case that will be considered in the rest of this article is to assume that the conformal factor is trivial (i. e. $\Omega = 1$) on the unphysical spacetime. This is somewhat of a strange simplification, because it requires that the spacetime M have empty boundary (since $\Omega^{-1}(0) = \partial M$)! One would thus not find oneself in this special case in practice since the whole point of the conformal constraint equations is to study hyperboloidal initial data in a conformally rescaled spacetime that has a boundary at future null infinity. Nevertheless, the simplification afforded by the assumption $\Omega = 1$ is worthwhile to consider from a mathematical point of view as preparation for handling situations where $\Omega \neq 1$, because the complexity of the equations is reduced considerably, yet many of their important features are preserved.

Substituting $\Omega = 1$ and $\Sigma = 0$ (which is consistent with the assumption that $\Omega = 1$ in spacetime since $\Sigma = n(\Omega)\big|_{\mathcal{Z}} = 0$ where n is the forward-pointing unit normal of $\mathcal{Z}$) into the equations (6) implies in addition that $L_{ab} = L_a = \psi = 0$ and thus reduces them to the following system of four coupled equations:

$$\begin{aligned} D_b \chi_{ac} - D_c \chi_{ab} &= S_{abc} \\ D^a S_{abc} &= \chi^a_b S_{ac} - \chi^a_c S_{ab} \\ D^a S_{ab} &= -\chi^{ac} S_{abc} \\ R_{ab} &= S_{ab} + \chi^c_c \chi_{ab} - \chi^c_a \chi_{cb} \, . \end{aligned} \tag{9}$$

Here, covariant derivatives are taken with respect to the induced metric h_{ab} of $\mathcal{Z}$ and χ_{ab} is the second fundamental form of $\mathcal{Z}$. As before, the tensor S_{ab} is symmetric and trace free with respect to h_{ab} whereas the tensor S_{abc} is antisymmetric in its last two slots, satisfies the Jacobi symmetry and is trace free with respect to h_{ab} on all its indices. These four quantities are the unknowns for which equations (9) must be solved. These equations will be called the *extended constraint equations*.

Notice that if the traces of the first and last equations equations are taken, then the usual constraint equations (7) result. Furthermore, if h_{ab} and χ_{ab} satisfy the usual constraint equations and one *defines* S_{abc} and S_{ab} by the first and last equations of (9) respectively, then the remaining two equations follow by straightforward algebra. Thus the extended constraint equations are fully equivalent to the usual constraint equations and solving the extended constraint equations can be seen as an alternative method for producing solutions of the usual constraint equations. This topic will be revisited later.

The equations (9) are clearly formally much simpler than the full system of conformal constraint equations. However, three essential features of the full equations still remain. The features in question concern the ellipticity of the various differential operators appearing in the equations. In order to see this, one must consider the principal symbols of the operators appearing on the left hand sides of the equations (9).

Recall that if $P : C^\infty(\mathbf{R}^n, \mathbf{R}^N) \to C^\infty(\mathbf{R}^n, \mathbf{R}^N)$ is a linear differential operator of order m, then it can be expressed as

$$P(u) = \sum_{\alpha_1 + \cdots + \alpha_n = m} \left(\sum_{i=1}^N b_i^{\alpha_1 \cdots \alpha_n} \frac{\partial^m u^i}{(x^1)^{\alpha_1} \cdots (x^n)^{\alpha_n}} \right) + P_0(u) \, ,$$

where P_0 is a differential operator of order less than or equal to $m-1$ and the $b_i^{\alpha_1 \cdots \alpha_n}$ are C^∞ functions. The principal symbol of P is the family of linear maps given by

$$\sigma_\xi(v) = \sum_{\alpha_1 + \cdots + \alpha_n = m} \left(\sum_{i=1}^N b_i^{\alpha_1 \cdots \alpha_n} \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} v^i \right)$$

for any non-zero $(\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$. The operator P is called *underdetermined elliptic* if the symbol is surjective for each non-zero ξ, *overdetermined elliptic* if the symbol is injective for each non-zero ξ and simply *elliptic* if the symbol is bijective for each non-zero ξ. The principal symbol of a nonlinear operator at a given $v \in C^\infty(\mathbf{R}^n, \mathbf{R}^N)$ is the principal symbol of the linearisation calculated at v. Notions of over- and underdeterminedness and ellipticity generalise accordingly.

The first equation in the extended constraint equations is linear in χ_{ab}. As an operator on χ_{ab}, its principal symbol is

$$\sigma_\xi : \chi_{ab} \mapsto \xi_b \chi_{ac} - \xi_c \chi_{ab} \,.$$

This linear operator has a one-dimensional kernel and is not surjective, as the following argument shows. Suppose that $\sigma_\xi(\chi_{ab}) = 0$. Since $\xi_a \xi^a \neq 0$, one can write uniquely $\chi_{ab} = \chi^0_{ab} + c\xi_a\xi_b$ for some c, where χ^0_{ab} is trace free. Substituting this expression for χ_{ab} yields

$$\xi_b \chi^0_{ac} - \xi_c \chi^0_{ab} = 0 \,. \tag{10}$$

Taking the trace over a and b implies that $\xi^c \chi^0_{ac} = 0$. Then, contracting with ξ^c gives $\xi^c \xi_c \chi^0_{ab} = 0$, or $\chi^0_{ab} = 0$. Consequently, the kernel of the symbol σ_ξ is one-dimensional, and consists of tensors of the form $c\xi_a\xi_b$. Next, since the space of symmetric 2-tensors is six-dimensional, the image of the symbol is five-dimensional. Since it can easily be verified that the target space of Jacobi tensors is eight-dimensional. The symbol can thus not be surjective. Note, however, that when it is restricted to trace free tensors, the principal symbol *is* injective. Consequently, the first equation of (9) is overdetermined elliptic when restricted to the space of trace free symmetric 2-tensors.

The second and third equations in (9) are linear in S_{abc} and S_{ab} respectively. It can be shown that they are underdetermined elliptic by showing that the principal symbols $S_{ab} \mapsto \xi^a S_{ab}$ and $S_{abc} \mapsto \xi^a S_{abc}$ are surjective maps from the five dimensional space of symmetric, trace free tensors onto the three dimensional space of 1-tensors and from the eight dimensional space of Jacobi tensors onto the three dimensional space of antisymmetric 2-tensors, respectively. These are fairly straightforward calculations and left to the reader.

The last equation for the metric h_{ab} is more subtle. It is quasilinear in h, with highest-order terms given by

$$h_{ab} \mapsto h^{cd}\left(\frac{\partial^2 h_{ad}}{\partial x^b \partial x^c} + \frac{\partial^2 h_{bd}}{\partial x^a \partial x^c} - \frac{1}{2}\frac{\partial^2 h_{ab}}{\partial x^c \partial x^d} - \frac{1}{2}\frac{\partial^2 h_{cd}}{\partial x^a \partial x^b} \right) .$$

The linearisation of this expression at a given metric is not formally either over- or underdetermined elliptic or elliptic. However, this phenomenon arises from the fact that the Ricci curvature is a geometric differential operator, and is thus invariant under changes of coordinates of the metric (also known as *gauge freedom*). Furthermore, if a specific coordinates are chosen (i. e. the gauge is fixed), then the Ricci curvature operator can be made formally elliptic in these coordinates. The standard choice (see, for example, [**?**]) is to use *harmonic coordinates* in which to express the Ricci curvature.

To understand what this entails, consider the Ricci curvature operator. Recall that if $\Gamma^a = h^{bc}\Gamma^a_{bc}$ (and also $\Gamma_a = h_{as}\Gamma^s$), where Γ^a_{bc} are the Christoffel symbols of the metric h_{ab}, then the components of the Ricci tensor satisfy

$$R_{ab} = R^H_{ab} + \frac{1}{2}(\Gamma_{a;b} + \Gamma_{b;a}) \tag{11}$$

where R^H_{ab} are the components of the *reduced* Ricci operator defined by

$$R^H_{ab} = -\frac{1}{2} h^{rs} h_{ab,rs} + q(\Gamma) \,. \tag{12}$$

Here, a comma denotes ordinary differentiation with respect to the coordinates, a semicolon denotes covariant differentiation (since Γ^a is not a tensor, this is to be taken formally; i. e. $\Gamma_{a;b} = \Gamma_{a,b} - \Gamma_s \Gamma^s_{ab}$), and $q(\Gamma)$ denotes a term that is quadratic in the components Γ^a. Coordinates functions x^a which are harmonic satisfy $\Delta_h x^a = 0$ for each a. It is easy to see that the metric components must then satisfy $\Gamma^a(h) = 0$. Consequently, $R_{ab}(h) = R^H_{ab}(h)$ in these coordinates; it is therefore clear from the definition of ellipticity that the Ricci operator is elliptic in h when h satisfies the harmonic coordinate condition.

The three features listed above — an operator whose symbol has a one-dimensional kernel, underdetermined operators and the operator whose ellipticity depends on the coordinates chosen — all appear in the full system of conformal constraint equations, as can be seen by inspecting the system (6), and will thus have to be dealt with eventually. The advantage provided by the assumption of trivial conformal factor is to end up with the extended constraint equations in which these features may be studied more or less in isolation. Techniques can thus be developed to deal with the subtleties of these features while remaining unencumbered by the complexity of the full system of conformal constraint equations.

Unfortunately, despite the simplifications outlined above, the extended constraint equations are still very complicated. In particular, it is not yet clear how to handle the $\chi_{ab} \mapsto D_b\chi_{ac} - D_c\chi_{ab}$ operator. However, the simplifications afforded by an additional assumption leads to an even more stripped-down system of equations in which this operator no longer appears. It is this new system which will be considered in the rest of this paper, and for which results have been obtained. A future paper by the Author will extend the methods below to handle the full system of extended constraint equations.

3 Asymptotically Flat Solutions of the Extended Constraint Equations in the Time Symmetric Case

3.1 Formulating an Elliptic Problem

The simplest case one can consider is to look for time-symmetric solutions of the extended constraint equations; that is, solutions for which $\chi \equiv 0$. Under this assumption, the equations further reduce to

$$\begin{aligned} D^a S_{ab} &= 0 \\ R_{ab}(h) &= S_{ab}\,, \end{aligned} \tag{13}$$

where S_{ab} is a symmetric tensor that is trace free with respect to the solution metric. One solution of these equations on the manifold $\mathcal{Z} = \mathbf{R}^3$ is clearly $h_{ab} = \delta_{ab}$, the Euclidean metric, and $S_{ab} = 0$. A natural, preliminary question in studying these equations is to investigate the space of solutions of metrics and symmetric trace free 2-tensors on $\mathbf{R}^3$ that are *near* $(\delta_{ab}, 0)$ in some suitable topology. Furthermore, attention will be restricted to *asymptotically flat* solutions of these equations — in other words, where $h_{ab} - \delta_{ab} \to 0$ in a suitable manner as the distance from the origin increases. The remainder of this article will properly formulate these notions and solve this problem.

Remark: Since (13) will be solved for metrics near the Euclidean metric, it is better to write metrics as small perturbations of the Euclidean metric of the form $\delta_{ab} + h_{ab}$. Thus (13) should be replaced with the system

$$\begin{aligned} D^a S_{ab} &= 0 \\ R_{ab}(\delta + h) &= S_{ab}\,, \end{aligned} \tag{14}$$

which is to be solved for h near 0 in some suitable topology. The covariant derivative here corresponds to the metric $\delta + h$.

The first difficulty to overcome is that the system of equations (14) fails to be elliptic at two levels, as discussed in the previous section: the operator $h \mapsto Ric(\delta + h)$ taking a suficiently small tensor h to the Ricci curvature of the metric $\delta + h$ is not elliptic unless appropriate coordinates are chosen; and the operator $S \mapsto \mathrm{div}(S)$ is underdetermined elliptic only when restricted to the space of trace-free symmetric tensors. The solvability of the system is thus very difficult to analyse directly since it seems to fall outside the realm of the standard theory of elliptic operators. The approach that will be taken for dealing with this problem is to modify the equations (14) somewhat in order to produce an elliptic system for which solvability is a much easier question, and then to show that solutions of the modified system actually solve the original system.

As suggested in the previous section, the Ricci curvature operator will be handled by posing the PDE problem in a harmonic coordinate system, thereby allowing the Ricci operator to be replaced by the reduced Ricci operator, which is indeed elliptic. The divergence term will be handled by a standard technique known as the York decomposition (see [**?**, **?**], originally [**?**]). Recall that the York splitting can be used to write a symmetric, trace-free tensor T as

$$T = T^* + \mathcal{L}^{\delta+h}(X) \, .$$

where T^* is the trace free and divergence free part of T, X is a 1-form, and $\mathcal{L}^{\delta+h}(X)$ is the conformal Killing operator with respect to the metric $\delta + h$ acting on X. This is defined for a general metic g by

$$\mathcal{L}^g_{ab}(X) = D_a X_b + D_b X_a - \frac{2}{3} D^c X_c g_{ab} \, ,$$

where D is the covariant derivative of the metric g. Furthermore, the composition of the divergence operator and the conformal Killing operator, that is the composite operator $\mathrm{div}_g \circ \mathcal{L}^g$ given componentwise by

$$X_a \mapsto D^a (D_a X_b + D_b X_a - \frac{2}{3} D^c X_c g_{ab}) = D^a D_a X_b + \frac{1}{3} D_b D^a X_a + R^s_b(g) X_s \, ,$$

is elliptic, as can easily be seen by computing its symbol.

These considerations lead to the following modification of the equations (14). The equations will be replaced by the elliptic system, given here in index-free notation for ease of presentation,

$$\begin{aligned} Ric^H(\delta + h) &= S(h, X, T) \\ \mathrm{div}_{\delta+h} \circ S(h, X, T) &= 0 \end{aligned} \tag{15}$$

where $S(h, X, T)$ will be called the York operator and is defined by

$$S(h, X, T) = T - \frac{1}{3} \mathrm{Trace}(T)(\delta + h) + \mathcal{L}^{\delta+h}(X) \tag{16}$$

for a 1-form X and *any* symmetric 2-tensor T (not necessarily one which is divergence and trace free). The equations (15) will be called the *reduced equations*. As will be shown in due course, the map defined by

$$\Phi(h, X, T) \equiv \big(Ric^H(\delta + h) - S(h, X, T), \mathrm{div}_{\delta+h} \circ S(h, X, T) \big) \tag{17}$$

on appropriate Banach spaces has a bounded, elliptic linearisation in the h and X directions. The space of solutions $\Phi^{-1}(0, 0)$ near $(0, 0, 0)$ can then be investigated using the tools of elliptic theory.

It is worthwhile to note at this stage that the method outlined above for solving (14) is in fact a method for solving the usual constraint equations in the time-symmetric case. Recall that the extended constraint equations imply the usual constraint equations by taking traces. Thus as one would expect, taking the trace of (14) leads exactly to the usual constraint equation $R(\delta + h) = 0$ under the time-symmetric assumption. Furthermore, it must be emphasised that this method of solving $R = 0$ is fundamentally different from the customary way in which this equation has been handled in the past. In the 'classical' method, which is due to Lichnerowicz and York (described in, for example, [**?**]), one prescribes a metric h_0 on $\mathbf{R}^3$ and considers the conformally rescaled metric $h = \exp u h_0$, where $u : \mathbf{R}^3 \to \mathbf{R}$ is an unknown function. One then reads the equation $R(\exp(u) h_0) = 0$ as a semi-linear elliptic equation for u. In contrast, the present method treats the metric h and the one-form X as the unknowns and leads to a quasi-linear elliptic system for these quantities. In the present case, the freely prescribable data will turn out to the quantity T.

3.2 Choosing the Banach Spaces

Before proceeding with the solution of the equations (15), it is necessary to specify in what Banach spaces of tensors the equations are to be solved. The notion of asymptotic flatness in $\mathbf{R}^3$ can be introduced rigorously by requiring that the relevant objects belong to a space of tensors with built-in control at infinity. Furthermore, the spaces should be chosen to exploit the ellipticity properties of the operators appearing in the map Φ. Both these ends are served by the choice of weighted Sobolev spaces. These are defined as follows.

Let T be any tensor on $\mathbf{R}^3$. (This tensor may be of any order — the norm $\|\cdot\|$ appearing in the following definition is then simply the norm on such tensors that is induced from the metric of $\mathbf{R}^3$.) The $H^{k,\beta}$ Sobolev norm of T is the quantity

$$\|T\|_{H^{k,\beta}} = \left(\sum_{l=0}^{k} \int_{\mathbf{R}^3} \|D^l T\|^2 \sigma^{-2(\beta-l)-3} \right)^{1/2} ,$$

where $\sigma(x) = (1+r^2)^{1/2}$ is the *weight function* and $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ is the distance to the origin. Note that Bartnik's conventions for describing the weighted spaces is being used (the reason for this is psychological: if $f \in H^{k,\beta}$ and f is smooth enough to invoke the Sobolev Embedding Theorem (see below), then $f(x) = O(r^\beta)$ as $r \to \infty$, which is easy to remember — see [**?**] for details). The space of $H^{k,\beta}$ functions of $\mathbf{R}^3$ will be denoted by $H^{k,\beta}(\mathbf{R}^3)$ and the space of $H^{k,\beta}$ sections of a tensor bundle B over $\mathbf{R}^3$ will be denoted by $H^{k,\beta}(B)$. As an abbreviation, or where the context makes the bundle clear, such a space may be indicated simply by $H^{k,\beta}$. Note also that the following convention for integration will be used in the rest of this paper. An integral of the form $\int_{\mathbf{R}^3} f$, as in the definition above, denotes an integral of f with respect to the standard Euclidean volume form. Integrals of quantities with respect to the volume form of a different metric will be indicated explicitly, as, for example, $\int_{\mathbf{R}^3} f \,\mathrm{dVol}_h$.

The spaces of $H^{k,\beta}$ tensors satisfy several important analytic properties and the reader is asked to consult Bartnik's paper, or others on the same topic [**?**, **?**, **?**, **?**], for details. The three most important properties that will be used in the sequel are the Sobolev Embedding Theorem, the Poincaré Inequality and Rellich's Lemma; these will be restated here for easy reference.

1. The Sobolev Embedding Theorem states that if $k > \frac{n}{2}$ and T is a tensor in $H^{k,\beta}$, then T is C^0. Furthermore, if the weighted C^k_β norm of a function f is given by

$$\|f\|_{C^k_\beta} = \sum_{l=0}^{k} \|D^l f \sigma^{-\beta+l}\|_0 ,$$

   where $\|f\|_0 = \sup\{|f(x)| \,:\, x \in \mathbf{R}^3\}$, then in fact, $T \in C^0_\beta$ and $\|T\|_{C^0_\beta} \leq C\|T\|_{H^{k,\beta}}$,

2. The Poincare Inequality states that if $\beta < 0$, then

$$\|f\|_{H^{0,\beta}} \leq C\|Df\|_{H^{0,\beta-1}} ,$$

   whenever f is a function in $H^{1,\beta}(\mathbf{R}^3)$.

3. The Rellich Lemma states that the inclusion $H^{k,\beta}(B) \subseteq H^{k',\beta'}(B)$, for any tensor bundle B, is compact when $k' < k$ and $\beta' > \beta$. In other words, if T_i is a uniformly bounded sequence of tensors in $H^{k,\beta}$, then there is a subsequence $T_{i'}$ converging to a tensor T in $H^{k',\beta'}$.

Remark: The constant C appearing in the estimates above is meant to depend only on the dimension n. In the remainder of this article, any constant depending only on n will be denoted by a generic C, unless it is important to emphasize otherwise.

Denote by $S_2(\mathbf{R}^3)$ the symmetric tensors over $\mathbf{R}^3$ (and denote by $\Lambda^1(\mathbf{R}^3)$ the 1-forms of $\mathbf{R}^3$). In order to ensure that a metric of the form $\delta + h$ is asymptotically flat and close to δ, one must choose

$h \in H^{k,\beta}(S_2(\mathbf{R}^3))$ for some $\beta < 0$. If k is sufficiently large (the actual choice will be made in the next section), the Sobolev Embedding Theorem then implies that $h_{ab}(r) = O(r^\beta)$ as $r \to \infty$, thus ensuring decay of h at infinity. The value of β can not be chosen to be too small, however. Since the Positive Mass Theorem [**?**] says that a non-trivial, asymptotically flat submanifold satisfying the constraint equations must have non-zero ADM mass, the r^{-1} term in the asymptotic expansion of h must be allowed to be non-zero, imposing the further requirement that $\beta > -1$.

This choice of Banach space for the metric imposes the choice of Banach space for the 1-forms X and tensors T. In fact, the only compatible choice defines Φ as an operator

$$\begin{gathered} H^{k,\beta}\big(S_2(\mathbf{R}^3)\big) \times H^{k-1,\beta-1}\big(\Lambda^1(\mathbf{R}^3)\big) \times H^{k-2,\beta-2}\big(S_2(\mathbf{R}^3)\big) \\ \Big\downarrow \Phi \\ H^{k-2,\beta-2}\big(S_2(\mathbf{R}^3)\big) \times H^{k-3,\beta-3}\big(\Lambda^1(\mathbf{R}^3)\big)\,. \end{gathered}$$

In other words, 1-forms X must be chosen in $H^{k-1,\beta-1}\big(\Lambda^1(\mathbf{R}^3)\big)$ and symmetric 2-tensors T must be chosen in $H^{k-2,\beta-2}\big(S_2(\mathbf{R}^3)\big)$. The reason for this is to ensure that the quantities $Ric^H(\delta+h) - S(h,X,T)$ and $\mathrm{div}_{\delta+h} \circ S(h,X,T)$ belong to weighted Sobolev spaces. Because of the differing number of derivatives taken on the h, X and T terms, this is so only when the domain spaces for the variables appearing there are chosen as indicated above. For instance, the reduced Ricci curvature operator is homogeneous and of degree two and thus sends a metric in $H^{k,\beta}$ to a tensor in $H^{k-2,\beta-2}$. The operator $S(h,X,T)$ is homogeneous but is only of degree one in X and of degree zero in T; it is thus in $H^{k-2,\beta-2}$ only when the weighting on X and T match together properly and match the weighting on the metric h as indicated above

3.3 Satisfying the Harmonic Coordinate Condition

Once the reduced equations (15) have been solved, one must then verify that the original equations (14) are satisfied by showing that the harmonic coordinate condition $\Gamma^a(\delta+h) = 0$ holds for the solution metric $\delta + h$. The purpose of the present section is to show that solutions of the reduced equations do indeed fulfill the harmonic coordinate condition automatically by virtue of the decay properties of the metric at infinity and an additional requirement on the degree of smoothness of the metric: that $h \in H^{k,\beta}$ with $k > \frac{7}{2}$. The question of solving the reduced equations will be dealt with in the next section.

Suppose that (h,X,T) is a solution of the reduced equations $\Phi(h,X,T) = (0,0)$ sufficiently near $(0,0,0)$. The vanishing of $\Gamma^a(\delta+h)$ can be established by applying a maximum principle to an equation satisfied by the quantity $\|\Gamma\|^2 = \Gamma^a\Gamma_a$. Write $h' = \delta + h$ for short. The Bianchi identity $\mathrm{div}_{h'}\big(Ric(h') - \frac{1}{2}R(h')h'\big) = 0$, applied to equation (11) defining the reduced Ricci operator yields the identity

$$0 = \big(\Gamma_{a;b} + \Gamma_{b;a} - \Gamma^c{}_{;c} h_{ab}\big)_{;}{}^{a} = \Gamma_{b;a}{}^{a} + R^a_b \Gamma_a\,,$$

after substituting the equation $\mathrm{div}_{h'} \circ Ric^H(h') = \mathrm{div}_{h'} \circ S(h',X,T) = 0$. One easily deduces

$$-\Delta_{h'}\|\Gamma\|^2 = 2\big(R_{ab}\Gamma^a\Gamma^b - \|D\Gamma\|^2\big)\,. \tag{18}$$

Since the metric h' is assumed to be close to δ, its Ricci curvature is thus small and so the right hand side of equation (18) is potentially negative. If it *were* negative but $\|\Gamma\|$ were non-zero and decaying at infinity, then the maximum principle would imply that $\|\Gamma\| = 0$, a contradiction. This line of argument can be made to work if one uses the properties of the weighted Sobolev space to which h belongs. A lemma concerning integration is required before beginning.

Duality Lemma: *If $u \in H^{l,\gamma}(\mathbf{R}^3)$ and $v \in H^{l-2,-\gamma-3}$, then the integral $\int_{\mathbf{R}^3} u \cdot v$ is well defined. Furthermore, the functional analytic dual space of $H^{0,\gamma}(\mathbf{R}^3)$ is isomorphic to $H^{0,-\gamma-3}(\mathbf{R}^3)$ under the pairing $v \mapsto \phi_v$ where $\phi_v(u) = \int_{\mathbf{R}^3} u \cdot v$.*

Proof: Choose u and v as in the statement of the lemma. Then by Hölder's inequality,

$$\begin{aligned}\int_{\mathbf{R}^3} |u \cdot v| &\leq \int_{\mathbf{R}^3} |u|\sigma^{-\gamma-3/2} \cdot |v|\sigma^{-(-\gamma-3)-3/2} \\ &\leq \left(\int_{\mathbf{R}^3} u^2\sigma^{-2\gamma-3}\right)^{1/2} \left(\int_{\mathbf{R}^3} v^2\sigma^{-2(-\gamma-3)-3}\right)^{1/2} \\ &< \infty\,.\end{aligned}$$

The product $u \cdot v$ is thus in L^1 and so its integral is well defined. The statement about duality follows from the Riesz Representation Theorem for L^2 and the inequality above. See [**?**, **?**] for details. □

If functions u and v are chosen such that $v \in H^{k,\gamma}(\mathbf{R}^3)$ and $u \in H^{k,-1-\gamma}(\mathbf{R}^3)$, then the integrals appearing in the left hand side of Green's identity for a general metric h on a large ball B_R, that is

$$\int_{B_R} u\Delta_h v \, \mathrm{dVol}_h + \int_{B_R} \nabla u \cdot \nabla v \, \mathrm{dVol}_h = \int_{\partial B_R} u \frac{\partial v}{\partial r} \, \mathrm{dA}_h\,, \tag{19}$$

where dA_h is the area form of the metric h, are all well defined as $R \to \infty$. Furthermore, Green's identity for C^∞ functions of compact support on $\mathbf{R}^3$ yields

$$\int_{B_R} u\Delta_h v \, \mathrm{dVol}_h + \int_{B_R} \nabla u \cdot \nabla v \, \mathrm{dVol}_h = 0$$

when R is large enough. Finally, since such functions are dense in $H^{k,\beta}$ spaces (see [**?**]), one can conclude that the limit of (19) as $R \to \infty$ is

$$\int_{\mathbf{R}^3} u\Delta_h v \, \mathrm{dVol}_h + \int_{\mathbf{R}^3} \nabla u \cdot \nabla v \, \mathrm{dVol}_h = 0\,.$$

With this in mind, integrate both sides of equation (18) against the volume for of the metric h' to obtain

$$-\frac{1}{2}\int_{\mathbf{R}^3} \Delta_{h'} \|\Gamma\|^2 \, \mathrm{dVol}_{h'} = \int_{\mathbf{R}^3} R_{ab}\Gamma^a\Gamma^b \, \mathrm{dVol}_{h'} - \int_{\mathbf{R}^3} \|D\Gamma\|^2 \, \mathrm{dVol}_{h'}\,. \tag{20}$$

Since $\Gamma \in H^{k-1,\beta-1}$, Green's Identity can be applied to the left hand side of (20) when $1 \in H^{k-1,-\beta}$. This is true since $\beta < 0$; thus the integral of the left hand side of (20) is zero. Consequently,

$$\begin{aligned}0 &\leq \int_{\mathbf{R}^3} \|Ric(h')\| \, \|\Gamma\|^2 \, \mathrm{dVol}_{h'} - \int_{\mathbf{R}^3} \|D\Gamma\|^2 \, \mathrm{dVol}_{h'} \\ &\leq \int_{\mathbf{R}^3} \|Ric(h')\| \, \|\Gamma\|^2 \, \mathrm{dVol}_{h'} - C\int_{\mathbf{R}^3} \|\, D\|\Gamma\| \,\|^2 \, \mathrm{dVol}_{h'}\end{aligned} \tag{21}$$

for some constant C, by the Cauchy-Schwartz inequality and straightforward algebra. Next, assume that h is small in a pointwise sense (this assumption follows without loss of generality: it follows from the Sobolev Embedding Theorem since h is sufficiently small in the $H^{k,\beta}$ norm and $k > \frac{3}{2}$). In fact, assume that h is sufficiently close to 0 so that all norms, derivatives and volume forms of the metric h' can be replaced by their Euclidean counterparts (at the expense of changing C of course). Since $\|\Gamma\|$ is a scalar function, the derivative operator in (21) can be replaced by the Euclidean derivative operator without introducing lower order terms. Thus, there exists a new constant C so that the estimate

$$0 \leq \int_{\mathbf{R}^3} \|Ric(h')\| \, \|\Gamma\|^2 - C\int_{\mathbf{R}^3} \|\, D\|\Gamma\| \,\|^2 \tag{22}$$

holds, where the norms and derivatives appearing here are those of the Euclidean metric. Next, $\mathrm{Ric}(h') \in H^{k-2,\beta-2}$ because $h' - \delta \in H^{k,\beta}$. But since $k > \frac{7}{2}$, the Sobolev Embedding Theorem gives $\mathrm{Ric}(h') \in C^0_{-\beta+2}$. That is,

$$\sup_{\mathbf{R}^3} \left\| \mathrm{Ric}(h') \cdot \sigma^{-\beta+2} \right\| \leq C < \infty \,,$$

which implies that

$$\sup_{\mathbf{R}^3} \left\| \mathrm{Ric}(h') \cdot \sigma^2 \right\| \leq C < \infty \,,$$

since $\beta < 0$. Finally, apply the Poincaré inequality for weighted Sobolev norms to $v = \|\Gamma\|$ to deduce

$$\begin{aligned}
\int_{\mathbf{R}^3} \|\mathrm{Ric}(h')\| \, \|\Gamma\|^2 &\leq \|\mathrm{Ric}(h') \cdot \sigma^2\|_0 \int_{\mathbf{R}^3} \|\Gamma\|^2 \sigma^{-2} \\
&\leq C \|\mathrm{Ric}(h')\|_{C^0_{-2}} \int_{\mathbf{R}^3} \| \, D\|\Gamma\| \, \|^2 \\
&\leq C \|h' - \delta\|_{C^2_0} \int_{\mathbf{R}^3} \| \, D\|\Gamma\| \, \|^2 \\
&\leq C \|h\|_{H^{k,\beta}} \int_{\mathbf{R}^3} \| \, D\|\Gamma\| \, \|^2 \qquad (23)
\end{aligned}$$

again by the Sobolev Embedding Theorem and the fact that $\beta < 0$. Using (23) in inequality (22) leads to the contradiction because the preceding estimates imply

$$0 \leq (C\|h\|_{H^{k,\beta}} - 1) \int_{\mathbf{R}^3} \| \, D\|\Gamma\| \, \|^2 \,,$$

while if $\|h\|_{H^{k,\beta}}$ is sufficiently small, the right hand side above is clearly negative. Avoiding this contradiction requires $D\|\Gamma\| = 0$. But since the Sobolev Embedding Theorem applied to $\Gamma \in H^{k-1,\beta-1}$ shows that $\|\Gamma\|$ decays at infinity when $\beta < 0$, it must be true that $\Gamma = 0$.

3.4 Attempting to Solve the Reduced Equations

The previous section shows that solutions of the reduced equations $\Phi(h, X, T) = (0, 0)$ in the chosen Banach spaces are solutions of the full equations. It thus remains only to solve the reduced equations. To this end, perturbative solutions of the reduced equations near $(0, 0, 0)$ will be found using the Implicit Function Theorem.

Implicit Function Theorem: *Let $\Phi : A \times B \to C$ be a C^1 map between Banach spaces and suppose that $\Phi(0,0) = 0$. If the restricted linearised operator $D\Phi(0,0)\big|_{\{0\}\times B} : B \to C$ is an isomorphism, then there exists an open set $\mathcal{U} \subset A$ containing 0 and a C^1 function $\phi : \mathcal{U} \to B$ with $\phi(0) = 0$ so that $\Phi\big(a, \phi(a)\big) = 0$.*

For an excellent discussion and proof of this theorem, see [**?**]. In order to use this theorem, the linearisation of the operator Φ at the origin must be calculated and its mapping properties understood.

The linearisation of Φ is actually quite simple when evaluated at the origin because the covariant derivative of the Euclidean metric is trivial. The only nonlinearities in Φ occur in the second order terms of the reduced Ricci operator and in terms that are quadratic in the derivatives of the metric (such as in products of Christoffel symbols or in the connection terms). It is thus easy to see that the linearisation of a covariant derivative operator at the Euclidean metric is just the Euclidean derivative operator, and it is a straightforward matter to deduce that

$$\mathrm{D}\Phi(0,0,0)\,(h, X, T) = \begin{pmatrix} -\frac{1}{2}\Delta h - \mathcal{L}(X) - T + \frac{1}{3}\mathrm{Tr}(T)\delta \\ \mathrm{div} \circ \big(\mathcal{L}(X) + T - \frac{1}{3}\mathrm{Tr}(T)\delta\big) \end{pmatrix} , \qquad (24)$$

where Δ is the Euclidean Laplacian and $\mathcal{L}$ is the Euclidean conformal Killing operator.

The Implicit Function Theorem asserts that if $D\Phi(\delta,0,0)\,(\cdot,\cdot,0)$ is a bounded linear operator and an isomorphism on $H^{k,\beta}(S_2(\mathbf{R}^3)) \times H^{k-1,\beta-1}(\Lambda^1(\mathbf{R}^3))$ (and Φ is a C^1 map of Banach spaces, of course), then the equation $\Phi(h,X,T)=(0,0)$ can be solved in the following sense. There are continuous functions

$$h : H^{k-2,\beta-2}(S_2(\mathbf{R}^3)) \to H^{k,\beta}(S_2(\mathbf{R}^3)) \qquad \text{and} \qquad X : H^{k-2,\beta-2}\big(S_2(\mathbf{R}^3)\big) \to H^{k-1,\beta-1}\big(\Lambda^1(\mathbf{R}^3)\big)$$

defined on neighbourhoods of 0 and mapping to neighbourhoods of 0, so that $\Phi\big(h(T),X(T),T\big) = (0,0)$.

Denote by P_δ the operator $D\Phi(0,0,0)(\cdot,\cdot,0)$. It is a bounded linear operator between the appropriate weighted Sobolev spaces because of the way in which the weights were chosen in Section 3.2. To determine whether P_δ is an isomorphism, one appeals to the following theorem. Its proof can be found in [**?**], but see also [**?**] for an excellent discussion of the intuitive foundation underlying the theory of elliptic operators on weighted spaces.

Invertibility Theorem: *Suppose B is any tensor bundle over $\mathbf{R}^3$ and let $Q : H^{k,\beta}(B) \to H^{k-2,\beta-2}(B)$ be any second order, elliptic, homogeneous, constant coefficient, partial differential operator mapping between weighted Sobolev spaces of sections of B, and $k \geq 2$. Then Q is surjective if $\beta > -1$ and injective if $\beta < 0$. It is thus bijective when $\beta \in (-1,0)$.*

Recall that the weight β in the domain spaces of P_δ must be chosen between -1 and 0 because of the considerations of Section 3.3. It remains to show whether P_δ is an isomorphism within this range of β.

Verify the injectivity of P_δ as follows. Solving the equations $P_\delta(h,X) = (0,0)$ amounts to finding h and X satisfying the equations

$$\begin{aligned} -\frac{1}{2}\Delta h + \mathcal{L}(X) &= 0 \\ \operatorname{div} \circ \mathcal{L}(X) &= 0\,. \end{aligned}$$

Since the operator $\operatorname{div}\circ\mathcal{L} : H^{k-1,\beta-1}(E) \to H^{k-3,\beta-3}(F)$ is an elliptic, homogeneous, constant coefficient operator of second order, the Invertibility Theorem applies, and since $\beta - 1 \in (-2,-1)$ when $\beta \in (-1,0)$, it is thus injective. Hence $X = 0$. The remaining equation now reads $\Delta h = 0$ and again, since $\Delta : H^{k,\beta}(E) \to H^{k-2,\beta-2}(F)$ and $\beta \in (-1,0)$, Δ is an isomorphism and thus is $h = 0$.

Although the operator P_δ is injective, the following argument shows that it fails to be surjective. First note that the Invertibility Theorem does not *guarantee* surjectivity in the same way that it guaranteed injectivity. To see this, attempt to solve the equations $P_\delta(h,X) = (f,g)$ for any $f \in H^{k-2,\beta-2}\big(S_2(\mathbf{R}^3)\big)$ and $g \in H^{k-3,\beta-3}\big(\Lambda^1(\mathbf{R}^3)\big)$. In other words, consider the system of equations

$$\begin{aligned} -\frac{1}{2}\Delta h - \mathcal{L}(X) &= f \\ \operatorname{div} \circ \mathcal{L}(X) &= g\,. \end{aligned}$$

Because $\beta - 1 \in (-2,-1)$, the operator $\operatorname{div}\circ\mathcal{L}$ is not necessarily surjective according to the Invertibility Theorem. The full equations $P_\delta(h,X) = (f,g)$ can thus not necessarily be solved.

To show that P_δ actually does fail to be surjective, it is necessary to show that the dimension of its cokernel in $H^{k,\beta}\big(S_2(\mathbf{R}^3)\big) \times H^{k-1,\beta-3}\big(\Lambda^1(\mathbf{R}^3)\big)$ is strictly greater that zero. First, note that if g is in the image of $\operatorname{div}\circ\mathcal{L}$, then the remaining equation $-\frac{1}{2}\Delta h = -\mathcal{L}(X) + f$ *can* be solved by the Invertibility Theorem since the weight β is chosen in the space in which Δ is an isomorphism. Thus the dimension of the cokernel of P_δ is equal to the dimension of the cokernel of $\operatorname{div}\circ\mathcal{L}$ as an operator between $H^{k-1,\beta-1}\big(\Lambda^1(\mathbf{R}^3)\big)$ and $H^{k-3,\beta-3}\big(\Lambda^1(\mathbf{R}^3)\big)$.

To characterise the cokernel of $\operatorname{div} \circ \mathcal{L}$, one appeals to general, function-theoretic properties of second order, homogeneous elliptic operators on weighted Sobolev spaces. The following theorem and its proof show how this is done.

Theorem: *Suppose B is any tensor bundle over $\mathbf{R}^3$ and let $Q : H^{k,\gamma}(B) \to H^{k-2,\gamma-2}(B)$ be a second order, homogeneous, elliptic operator with constant coefficients mapping between weighted Sobolev spaces of sections of B where $k \geq 2$ and $\gamma < -1$. The operator Q is not surjective and its image is the space:*

$$\operatorname{Im}(Q) = \left\{ f \in H^{k-2,\gamma-2}(B) \;:\; \int_{\mathbf{R}^3} \langle f, g \rangle = 0 \quad \forall \quad g \in \operatorname{Ker}\left(Q^*; -1-\gamma\right)\right\}, \tag{25}$$

where the inner product $\langle \cdot, \cdot \rangle$ is induced on B from the Euclidean metric of $\mathbf{R}^3$, the operator Q^* is the formal adjoint of Q, and $\operatorname{Ker}(Q^*; -1-\gamma)$ is its kernel as an operator from $H^{k,-1-\gamma}(B)$ to $H^{k-2,-3-\gamma}(B)$.

Proof: Denote the space on the right hand side of equation (25) by A. Suppose that $k = 2$ and consider first the containment $\operatorname{Im}(Q) \subseteq A$. Choose $Q(x) \in \operatorname{Im}(Q)$ and $g \in H^{2,-1-\gamma}(B)$ so that $Q^*(g) = 0$. Since $Q(x) \in H^{2,\gamma-2}(B)$, the integral $\int_{\mathbf{R}^3} \langle Q(x), g \rangle$ is well defined by the Duality Lemma and equals $\int_{\mathbf{R}^3} \langle x, Q^*(g) \rangle$ by definition of the adjoint. The integral is thus zero and so $Q(x) \in A$.

The reverse containment $A \subseteq \operatorname{Im}(Q)$ is proved as follows. Suppose f belongs to A; in other words, $f \in H^{0,\gamma-2}(B)$ and satisfies $\int_{R^3} \langle f, g \rangle = 0$ for all $g \in H^{k-1,-1-\gamma}(B)$ with $Q^*(g) = 0$. Suppose also that $f \notin \operatorname{Im}(Q)$. Since Q is elliptic, $\operatorname{Im}(Q)$ is closed; thus by the Hahn-Banach theorem, there exists a linear functional ϕ on $H^{0,\gamma-2}(B)$ so that $\phi(f) \neq 0$ but $\phi\big|_{\operatorname{Im}(Q)} = 0$. Again by the Duality Lemma, there is a unique $g \in H^{0,-1-\gamma}(B)$ so that $\phi(f) = \int_{\mathbf{R}^3} \langle f, g \rangle$ for all $f \in H^{0,\gamma-2}(B)$. But now, $\phi\big|_{\operatorname{Im}(Q)} = 0$ implies that

$$\begin{aligned} 0 &= \phi(Q(T)) \\ &= \int_{\mathbf{R}^3} \langle g, Q(T) \rangle \\ &= \int_{\mathbf{R}^3} \langle Q^*(g), T \rangle \end{aligned}$$

for all $T \in H^{2,\gamma}(B)$. Thus $Q^*(g) = 0$ or $g \in \operatorname{Ker}(Q^*; -1-\gamma)$. But now, the assumptions $\phi(f) \neq 0$ and $\int_{\mathbf{R}^3} \langle f, g \rangle = 0$ for all $g \in \operatorname{Ker}(Q^*; -1-\gamma)$ are mutually contradictory. Thus it must be that $f \in \operatorname{Im}(Q)$. Finally, the analogous result for $k > 2$ follows similarly, and uses elliptic regularity theory. $\square$

Apply this theorem to the operator $Q = \operatorname{div} \circ \mathcal{L}$ with $\gamma = \beta - 1$. Now, $Q^* = Q$, so in order to solve the equation $\operatorname{div} \circ \mathcal{L}(X) = g$, the tensors g must satisfy the constraints

$$\int_{\mathbf{R}^3} g_a Y^a = 0\,,$$

where Y is any tensor in the kernel of the operator Q in the space $H^{k-1,-1-\gamma}\big(\Lambda^1(\mathbf{R}^3)\big)$.

The kernel of $Q = \operatorname{div} \circ \mathcal{L}$ is well known and consists of 1-forms dual to the the conformal Killing fields of $\mathbf{R}^3$. There are precisely ten linearly independent such vector fields: the translation vector fields, the rotation vector fields, the dilation field and three so-called *special* conformal Killing fields (these correspond to transformations of the form $i \circ T \circ i$, where i is the inversion with respect to the unit circle and T is a translation). The asymptotic behaviour of these vector fields can thus be computed exactly: the translations have constant norm, the rotations and dilations have norm growing linearly in the distance from the origin, and the special vector fields have quadratic growth in the distance from the origin. Since $-1-\gamma \in (0,1)$ when $\beta \in (-1,0)$, the only 1-forms dual to the conformal Killing fields in $H^{k-1,-1-\gamma}\big(\Lambda^1(\mathbf{R}^3)\big)$ are thus the translation 1-forms $\mathrm{d}x^1$, $\mathrm{d}x^2$ and

$\mathrm{d}x^3$. Consequently, the image of $Q = \operatorname{div} \circ \mathcal{L}$ in the space $H^{k-3,\gamma-2}\big(\Lambda^1(\mathbf{R}^3)\big)$ can be characterised as follows:

$$\operatorname{Im}(Q) = \left\{ g \in H^{k-3,\beta-3}\big(\Lambda^1(\mathbf{R}^3)\big) \ : \ \int_{\mathbf{R}^3} g_a = 0\,,\ a = 1, \ldots, 3 \right\} ,$$

where g_a are the components of g in the standard coordinates of $\mathbf{R}^3$.

The conclusion that can be drawn from the preceding analysis is that the equation $\Phi(h, X, T) = (0, 0)$ is *not* solvable near $(0, 0, 0)$ using the Implicit Function Theorem. The non-surjectivity of the linearised operator at $(0, 0, 0)$ is the essential obstruction. The best that can be achieved using the Implicit Function Theorem is that the equation $\Phi(h, X, T) = (0, 0)$ can be solved *up to* a term that is transverse to the space $\operatorname{Im}(\operatorname{div} \circ \mathcal{L})$. It will turn out, that this is nevertheless sufficient for solving the full equations.

3.5 Reestablishing Surjectivity and Solving the Reduced Equations

As indicated in the Section 3.4, the operator P_δ is not surjective onto its target space. However, because the image of this operator has finite codimension and can be characterised in terms of integrability conditions, it is possible to construct a differential operator Φ' that is closely related to Φ, whose linearisation *is* surjective (and remains injective). Thus it will be possible to solve the related problem $\Phi'(h, X, T) = (0, 0)$ instead of the original problem. Of course, the question of satisfying the harmonic coordinate condition must now be revisted, and this will be discussed in the next section.

In order to proceed, first note that $H^{k-3,\beta-3}\big(\Lambda^1(\mathbf{R}^3)\big)$ can be written as $\operatorname{Im}(\operatorname{div} \circ \mathcal{L}) \oplus C$ in many different ways; but in each case, C is a three dimensional subspace of $H^{k-3,\beta-3}\big(\Lambda^1(\mathbf{R}^3)\big)$ whose members do not integrate to zero upon taking the Euclidean inner product with the translation 1-forms. One such choice is

$$C = \operatorname{span}\left\{\phi \, \mathrm{d}x^a\right\}_{a=1,2,3} ,$$

where ϕ is any positive, C_c^∞ function whose integral over $\mathbf{R}^3$ is equal to 1.

Denote by B the domain space of the operator Φ. The previous paragraph suggests that one should attempt to construct an operator Φ' that extends Φ in such a way that $\Phi' : B \times \mathbf{R}^3 \to \operatorname{Im}(P_\delta) \oplus C$, where the additional $\mathbf{R}^3$ factor in the domain should map under the linearisation $\mathrm{D}\Phi'$ at the solution $(0, 0, 0; 0) \in B \times \mathbf{R}^3$ onto the C factor in the image. If such a construction is possible, then the equation $\Phi'(h, X, T; \lambda) = (0, 0)$ can be solved using the Implicit Function Theorem.

Construct the operator $\Phi' : B \times \mathbf{R}^3 \to H^{k-3,\beta-3}\big(\Lambda^1(\mathbf{R}^3)\big)$ according to the prescription

$$\Phi'(h, X, T; \lambda) = \left(Ric^H(\delta + h) - S(h, X, T),\, \operatorname{div}_{\delta+h} \circ S(h, X, T) - \sum_{a=1}^{3} \lambda_a \phi \, \mathrm{d}x^a \right) , \tag{26}$$

where, as before, Ric^H is the reduced Ricci operator and $S(\cdot, \cdot, \cdot)$ is the York operator. The linearisation of Φ' at $(0, 0, 0; 0)$ in the directions transverse to the T direction is easily seen to be

$$\mathrm{D}\Phi'(\delta, 0, 0; 0)(h, X, 0, \lambda) = \left(-\frac{1}{2}\Delta h - \mathcal{L}(X),\, \operatorname{div} \circ \mathcal{L}(X) - \sum_{a=1}^{3} \lambda_a \phi \, \mathrm{d}x^a \right) . \tag{27}$$

This new operator is still bounded because ϕ has compact support. The modification introduced by the extra $\mathbf{R}^3$ factor is enough to make this operator bijective, as the following arguments show.

First, suppose $\mathrm{D}\Phi'(0, 0, 0; 0)(h, X, 0, \lambda) = (0, 0)$. Integrate the components of the second equation; by the divergence theorem for the Euclidean metric (valid because constant functions can be integrated against $H^{k-3,\beta-3}$ functions when $\beta \in (-1, 0)$ according to the Duality Lemma), the divergence terms integrate to zero, yielding $\lambda_a = 0$ for all a. The argument that both X and h are then equal to zero follows as in Section 3.4.

Next, attempt to solve the equations $\mathrm{D}\Phi'(0,0,0;0)(h,X,0,\lambda)=(f,g)$. First choose the components λ_a so that

$$\int_{\mathbf{R}^3}\big(g_a+\lambda_a\phi\big)=0$$

for each a. The equation $\operatorname{div}\circ\mathcal{L}(X)=g-\sum_{a=1}^{3}\lambda_a\phi\,\mathrm{d}x^a$ can then be solved for X_g according to the characterisation of the image of the operator $\operatorname{div}\circ\mathcal{L}$ from Section 3.4. The remaining equation $-\frac{1}{2}\Delta h=-\mathcal{L}(X_g)+f$ can then be solved because $\beta\in(-1,0)$ makes Δ an isomorphism.

The Implicit Function Theorem can now be invoked to solve the equation $\Phi'(h,X,T;\lambda)=(0,0)$ near $(0,0,0;0)$. To be precise, there is a neighbourhood $\mathcal{U}\subset H^{k-2,\beta-2}\big(S^2(\mathbf{R}^3)\big)$ with the following property. If $T\in\mathcal{U}$, then there is a metric $\delta+h(T)$ with $h(T)\in H^{k,\beta}\big(S^2(\mathbf{R}^3)\big)$, a covector field $X(T)\in H^{k-1,\beta-1}\big(\Lambda^1(\mathbf{R}^3)\big)$, and three real numbers $\lambda_a(T)$ so that $\Phi'\big(h(T),X(T),T;\lambda(T)\big)=(0,0)$. Furthermore, the various functions $T\mapsto h(T)$, etc. are C^1 in the appropriate Banach space norms. In other words, there exists a constant C so that

$$\begin{aligned}\|h\|_{H^{k,\beta}}&\le C\|T\|_{H^{k-2,\beta-2}}\\ \|X\|_{H^{k-1,\beta-1}}&\le C\|T\|_{H^{k-2,\beta-2}}\\ \|\lambda\|_{\mathbf{R}^3}&\le C\|T\|_{H^{k-2,\beta-2}}\,,\end{aligned}\tag{28}$$

where $\|\cdot\|_{\mathbf{R}^3}$ denotes the standard Euclidean norm of $\mathbf{R}^3$, as long as $T\in\mathcal{U}$.

3.6 There is a Such Thing as a Free Lunch

Section 3.5 shows how the reduced equations (15) can be modified in such a way that they can be solved using the Implicit Function Theorem. This procedure results in a family of solutions of the equations

$$\begin{aligned}Ric^H(\delta+h)&=S(h,X,T)\\ \operatorname{div}_{\delta+h}\circ S(h,X,T)&=\lambda\phi\,,\end{aligned}\tag{29}$$

where $\lambda=\sum_{a=1}^{3}\lambda_a\,\mathrm{d}x^a$. It remains to show whether this technique allows any headway to be made in solving the original equations (14). In other words, it must be shown that a solution with $\lambda=0$ can be found and that the harmonic coordinate condition can be satisfied.

The structure of the equations (29) will actually *force* $\lambda=0$. To see this, assume that both λ and the quantities Γ^a are nonzero. As before, the Bianchi identity can be used to show that if $(h,X,T;\lambda)$ solves $\Phi'(h,X,T;\lambda)=(0,0)$, then Γ satisfies the equation

$$\Delta_{\delta+h}\Gamma_a+R_a^b\Gamma_b=2\phi\lambda_a\tag{30}$$

rather than equation (18) as in Section 3.3. It is tempting now to mimic the remainder of the argument of Section 3.3 and conclude that Γ must vanish due to the maximum principle, thereby forcing λ to vanish as well. This is problematic, however, because the estimates used there fail here to lead to a contradiction when $\lambda\neq0$. It is thus necessary to proceed with a less direct argument. If Q_h denotes the operator $u_a\mapsto\Delta_{\delta+h}u_a+[Ric(\delta+h)]_a^b u_b$, then (30) asserts that $2\phi\lambda_a$ is in the image of $H^{k,\beta-1}(\Lambda^1(\mathbf{R}^3))$ under Q_h, where $\beta\in(-1,0)$. This, however, is impossible according to the following argument.

The calculation Section 3.3 actually shows that the operator Q_h acting on $H^{l,\gamma}$ 1-forms is injective for all $\gamma<-1$ whenever h is sufficiently close to zero in the $H^{k,\beta}$ norm. Furthermore, the additional term in (29) was specifically chosen to satisfy the integrability condition $\int_{\mathbf{R}^3}\langle\lambda\phi,\mathrm{d}x^b\rangle\neq0$ (since $\lambda_a=0$ for all a). This condition ensures that $2\phi\lambda_a$ is not in the image of the operator $Q_0=\Delta_\delta$ acting on the space of $H^{l,\gamma}$ 1-forms of $\mathbf{R}^3$ (because the the image of Δ_δ in $H^{l,\gamma}$ for $\gamma<-1$ is perpendicular to the harmonic polynomials of degree less than the nearest integer less than γ, and this always includes the constants). The following situation thus presents itself: Q_h

is a family of injective, elliptic operators and $2\phi\lambda$ is not in the image of Q_0. One would like to conclude that $2\phi\lambda$ is not in the image of any Q_h either, provided h is sufficiently close to 0. This contradiction would indicate that indeed, $\lambda = 0$; and then the argument of Section 3.3 could be used to conclude that $\Gamma = 0$ as well.

In order to complete the proof of the Main Theorem, it thus remains only to show that $\phi\lambda$ does not belong to the image of any Q_h when h is sufficiently close to 0 in the $H^{k,\beta}$ norm. But this result is essentialy due to the stability of kernels and cokernels of general elliptic operators and can be seen by applying the following lemma.

Stability Lemma: *Let B be a tensor bundle over $\mathbf{R}^3$ and let $Q_\varepsilon : H^{l,\gamma}(B) \rightarrow H^{l-2,\gamma-2}(B)$, $\varepsilon \in [0,1]$, be a continuous family of linear, homogenous, second order, elliptic operators, for all $\gamma < -1$. Furthermore, suppose Q_ε is uniformly injective for any ε whenever $\gamma < -1$; i. e. for each $\gamma < -1$, there is a constant C independent of ε so that $\|Q_\varepsilon(T)\|_{H^{l-2,\gamma-2}} \geq C\|T\|_{H^{k,\gamma}}$. If $Y \notin \mathrm{Im}(Q_0)$, then there exists $\varepsilon_0 > 0$ so that $Y \notin \mathrm{Im}(Q_\varepsilon)$ for all $\varepsilon < \varepsilon_0$.*

Proof: Suppose the contrary; then for some $\gamma < -1$, there exists a sequence $\varepsilon_i \rightarrow 0$ and a sequence $T_i \in H^{l,\gamma}(B)$ so that $Y = Q_{\varepsilon_i}(T_i)$. By the uniform injectivity of Q_ε, $\|T_i\|_{H^{l,\gamma}} \leq C\|Y\|_{H^{l-2,\gamma-2}}$ and is thus uniformly bounded. By Rellich's Lemma, there exists a subsequence $T_{i'}$ which converges to an element T in $H^{l-1,\gamma+\rho}$, where ρ is small enough so that $\gamma + \rho < -1$. Again, by uniform injectivity,

$$\begin{aligned}
\|T_{i'} - T_{j'}\|_{H^{l,\gamma+\rho}} &\leq C\|Q_{\varepsilon_{i'}}(T_{i'} - T_{j'})\|_{H^{l-2,\gamma+\rho-2}} \\
&\leq C\|(Q_{i'} - Q_{j'})(T_{j'})\|_{H^{l-2,\gamma+\rho-2}} \\
&\leq C\|Q_{i'} - Q_{j'}\|_{op} \cdot \|T_{j'}\|_{H^{l,\gamma+\rho}} \\
&\leq C\|Q_{i'} - Q_{j'}\|_{op} \cdot \|T_{j'}\|_{H^{l,\gamma}} \\
&\longrightarrow 0\,,
\end{aligned}$$

by the continuity of Q_ε and the uniform boundedness of T_i. Here, $\|\cdot\|_{op}$ denotes the relevant operator norm. The subsequence $T_{i'}$ is thus Cauchy in the $H^{l,\gamma+\rho}$ norm and so $T_{i'} \rightarrow T$ in this norm. But now,

$$Y = \lim_{i' \rightarrow \infty} Q_{\varepsilon_{i'}}(T_{i'}) = Q_0(T)\,,$$

contradicting the fact that $Y \notin \mathrm{Im}(Q_0)$. □

This lemma can be applied to the operators $Q_\varepsilon = Q_{\varepsilon h}$ by noting that uniform injectivity follows in the standard way from the injectivity of each Q_h and the fact that the constant in the elliptic estimate for these operators is independent of h, provided h is sufficiently near to 0. As outlined above, the contradiction engendered by this lemma completes the proof of the Main Theorem.